GREEN CURRENTS FOR QUASI-ALGEBRAICALLY STABLE MEROMORPHIC SELF-MAPS OF $\mathbb{P}^k$

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Abstract. We construct a canonical Green current T_f for every quasi-algebraically stable meromorphic self-map f of $\mathbb{P}^k$ whose first dynamical degree is strictly greater than 1. We establish functional equations for T_f and show that the support of T_f is contained in the Julia set, which is thus non empty.

1. Introduction

Let $f : \mathbb{P}^k \longrightarrow \mathbb{P}^k$ be a meromorphic self-map. It can be written $f = [G_0 : \ldots : G_k]$ in homogeneous coordinates where the G_j's are homogeneous polynomials in the $k+1$ variables $z_0, \ldots, z_k$ of the same degree d with no nontrivial common factor. The polynomial $F := (G_0, \ldots, G_k)$ will be called *a lifting* of f in $\mathbb{C}^{k+1}$. The number $\mathrm{d}(f) := d$ will be called *the algebraic degree* of f. Moreover, f is said to be *dominant* if it is generically of maximal rank k, in other words, its jacobian determinant does not vanish identically (in any local chart). From now on, we always consider dominant meromorphic self-maps f of $\mathbb{P}^k$ with $k \geq 2$. For $n \in \mathbb{N}$, f^n denotes $f \circ \cdots \circ f$ (n times).

Recall the following definition (see [13, 14, 17])

Definition 1. *A meromorphic self-map $f : \mathbb{P}^k \longrightarrow \mathbb{P}^k$ is said to be* algebraically stable *(or* AS *for short) if* $\mathrm{d}(f^n) = \mathrm{d}(f)^n$, $n \in \mathbb{N}$.

In other words, f is AS if and only if a sequence $(F_n)_{n=1}^{\infty}$ of liftings of $(f^n)_{n=1}^{\infty}$ can be defined as follows

$$F_n := F_1 \circ F_{n-1}, \qquad n \geq 1, \tag{1.1}$$

where $F_1,\ F_0$ are arbitrarily fixed liftings of $f,\ f^0 := \mathrm{Id}$ respectively.

For every AS map f with $\mathrm{d}(f) > 1$, Sibony proves in [17] that the following limit in the sense of current

$$T := \lim_{n\to\infty} \frac{(f^n)^*\omega}{\mathrm{d}(f^n)} \tag{1.2}$$

exists, where ω denotes the Fubiny-Study Kähler form on $\mathbb{P}^k$ so normalized that $\int_{\mathbb{P}^k} \omega^k = 1$. T is called *the Green current* associated to f. He also proves that T does not charge any hypersurfaces. Given a positive integer d, a "generic" meromorphic self-map of algebraic degree d is always AS (see [14]). In the last decades the study

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of Green currents plays a central role in Complex Dynamics in higher dimensions. We address the reader to the classical works of Fornæss-Sibony [13, 14], Sibony [17] for further explanations. Newer articles on the topic could be found in [7, 9, 11, 15] etc.

In contrast to the case of AS maps, the dynamics of non AS maps are very poorly understood. This question remains almost unexplored up to now. Two fundamental problems arise:

Problem 1. Study the degree growth of non AS maps.
Problem 2. Define a natural Green currents for such maps.

One of the first works in this direction is the article of Bonifant–Fornæss [5] where some special non AS maps are thoroughly studied. In her thesis [4] Bonifant constructs an appropriate Green current for these maps and then writes down the functional equation. J. Diller and Ch. Favre (see [6]) have constructed Green currents for birational maps of compact Kähler surface. In the case of polynomial maps of $\mathbb{C}^2$, the problem has also been investigated by Ch. Favre and M. Jonsson (see [12]). Moreover, S. Boucksom, Ch. Favre and M. Jonsson have studied the degree growth for meromorphic surface maps (see [3]). In higher dimension the situation is completely open, and the two questions above remain a great difficulty. There are series of interesting examples of birational maps acting on the space of complex square matrices which were worked out by E. Bedford and K. Kim (see [1, 2] etc). However, a general theoretical approach remains to be built.

In this paper we study a new class of non AS self-maps of $\mathbb{P}^k$. We construct the good Green currents for them, write down the functional equations and show that the support of the Green current is contained in the Julia set of the corresponding self-map. Here is the formal definition of the new class.

Definition 2. *A meromorphic self-map f of $\mathbb{P}^k$ is said to be* quasi-algebraically stable *(or* QAS *for short) if a sequence of liftings $(F_n)_{n>n_0}^{\infty}$ for the iterates $(f^n)_{n>n_0}^{\infty}$ can be defined as follows*

$$F_n := \frac{F_1 \circ F_{n-1}}{H \circ F_{n-n_0-1}}$$

for all $n > n_0$. Here $n_0 \geq 1$ is an integer, H is a homogeneous polynomial of $k+1$ variables, and $F_0, \ldots, F_{n_0}$ are arbitrarily fixed liftings of $f^0 = \mathrm{Id}, \ldots, f^{n_0}$.

It is worthy to compare the above recurrent formula with (1.1). In the previous work [16] the author has introduced a criterion in order to test if a non AS self-map is QAS (see Condition (i)–(iii) in Theorem 4.4 below [1]). As it was shown in [16] there are a lot of non AS self-maps which are QAS. This speculation will be reconfirmed in the present work where a new family of QAS self-maps in $\mathbb{P}^2$ is exhibited.

This paper is organized as follows.

We begin Section 2 by collecting some background and introducing some notation. This preparatory is necessary for us to state the main theorem afterwards.

[1] In fact, in that paper all self-maps which satisfy the latter criterion were called QAS. But by now we would like to take Definition 2 as the new definition for QAS maps. This will enlarge the class of QAS self-maps.

Section 3 is devoted to the proof of the main theorem.

Finally, Section 4 concludes the paper with a new family of QAS self-maps in $\mathbb{P}^2$.

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2. Statement of the main result

First we fix some notation and terminology.

2.1. Meromorphic self-maps and positive closed currents of bidegree $(1,1)$. Let f be a meromorphic self-map of $\mathbb{P}^k$. The *indeterminacy locus* $\mathcal{I}(f)$ of f is the set of all points of $\mathbb{P}^k$ where f is not holomorphic, or equivalently the common zero set of $k+1$ component polynomials $G_0,\ldots,G_k$, where $(G_0,\ldots,G_k)$ is a lifting of f. Observe that $\mathcal{I}(f)$ is a subvariety of codimension at least 2. The *first dynamical degree* of f, denoted by $\lambda_1(f)$, is given by

$$\lambda_1(f) := \lim_{n\to\infty} \mathrm{d}(f^n)^{\frac{1}{n}}. \tag{2.1}$$

For a discussion on dynamical degrees of meromorphic self-maps see the articles of Dinh–Sibony [8, 10]. We denote by $\mathcal{C}^+(\mathbb{P}^k)$ the set of positive closed currents of bidegree $(1,1)$ on $\mathbb{P}^k$. The mass of T is defined by $\|T\| := \int\limits_{\mathbb{P}^k} T\wedge\omega^{k-1}$. We consider the cone $\mathcal{P}$ of plurisubharmonic functions u in $\mathbb{C}^{k+1}$, satisfying the following homogeneity property: there exist $c>0$ such that if $\lambda\in\mathbb{C}$, then

$$u(\lambda z) = c\log|\lambda| + u(z), \qquad z\in\mathbb{C}^{k+1}.$$

The functions in $\mathcal{P}$ are so normalized that $\sup\limits_{\mathbb{B}} u = 0$, where $\mathbb{B}$ denotes the unit ball in $\mathbb{C}^{k+1}$.

With a function u satisfying the above homogeneity property (but not necessarily the normalization condition), we are going to associate a current $T\in\mathcal{C}^+(\mathbb{P}^k)$. Let $\pi:\ \mathbb{C}^{k+1}\setminus\{0\}\to\mathbb{P}^k$ be the canonical projection. Let U be an open set in $\mathbb{P}^k$ such that there is a holomorphic inverse $s:\ U\to\mathbb{C}^{k+1}\setminus\{0\}$ of π, that is, $\pi\circ s = \mathrm{Id}$. Define T on U by $T := \mathrm{dd^c}(u\circ s)$. Then T is independent of s. With this local definition, we have an operator $\mathcal{L}$ defined by $\mathcal{L}(u) := T$. It is well-known (see [13, 17]) that $\mathcal{L}$ is an isomorphism between $\mathcal{P}$ and $\mathcal{C}^+(\mathbb{P}^k)$. Moreover, if $T=\mathcal{L}(u)$ and $u(\lambda z) = c\log|\lambda| + u(z)$ for $z\in\mathbb{C}^{k+1},\ \lambda\in\mathbb{C}$, then $\|T\| = c$.

Any (not necessarily reduced) hypersurface $\mathcal{H}$ of $\mathbb{P}^k$ defines a current of integration $[\mathcal{H}] = [H=0]\in\mathcal{C}_1^+(\mathbb{P}^k)$, where $H:\ \mathbb{C}^{k+1}\longrightarrow\mathbb{C}$ is a homogeneous polynomial defining $\mathcal{H}$. Moreover,

$$\|[H=0]\| = \deg(H), \tag{2.2}$$

where $\deg(H)$ is the (homogeneous) degree of H.

For a current $T\in\mathcal{C}^+(\mathbb{P}^k)$, we define the pull-back of T by f as follows

$$f^*T := \mathcal{L}(u\circ F),$$

where $u := \mathcal{L}^{-1}(T)$ and F is a lifting of f. It is easy to see that

$$\|f^*T\| = \mathrm{d}(f) \cdot \|T\|. \tag{2.3}$$

For further information on this matter, the reader is invited to consult the works [13] and [17]. Finally, for a function $u : \mathbb{C}^{k+1} \to [-\infty, \infty]$, let u^* denote its *upper semicontinuous regularization.*

2.2. **Statement of the main result.** Now we are ready to formulate the following

Main Theorem. *Let f be a QAS self-map as in Definition 2. Suppose in addition that $\lambda_1(f) > 1$.*

(i) *Then $\big(\limsup\limits_{n\to\infty} \frac{\log\|F_n\|}{\mathrm{d}(f^n)}\big)^*$ exists and defines a plurisubharmonic function u in $\mathbb{C}^{k+1}$.*

(ii) *Let $T := \mathcal{L}(u)$. Then the following functional equation holds*

$$f^*(T) = \lambda_1(f) \cdot T + \frac{\mathrm{d}(f) - \lambda_1(f)}{\deg(H)} \cdot [H = 0].$$

(iii) *The support of the current T is contained in the Julia set, which is thus non empty.*

It is worthy to remark here that the presence of a factor $[H = 0]$ in the functional equation (ii) characterizes QAS self-maps. Indeed, for an AS map f with corresponding Green current T (see (1.2)), the functional equation is $f^*T = \mathrm{d}(f) \cdot T$. In Section 3 below we develop necessary preparatory results and prove the Main Theorem. Section 4 is devoted to a new family of examples.

3. Proof

In this section we keep the hypothesis of the Main Theorem and the notation of Section 2. To simplify the exposition, from now on we will write λ, d and h instead of $\lambda_1(f)$, $\mathrm{d}(f)$ and $\deg(H)$ respectively. As an immediate consequence of Definition 2, we get

$$\mathrm{d}(f^n) = \begin{cases} \mathrm{d}(f)^n, & n = 0, \dots, n_0, \\ \mathrm{d}(f) \cdot \mathrm{d}(f^{n-1}) - h \cdot \mathrm{d}(f^{n-n_0-1}), & n > n_0. \end{cases} \tag{3.1}$$

On the other hand, it has been shown in [16, Theorem 4.2] that λ is the root of maximal modulus of the characteristic polynomial

$$P(t) = t^{n_0+1} - dt^{n_0} + h. \tag{3.2}$$

Let r be the multiplicity of λ.

Lemma 3.1. *1) There exist $Q \in \mathbb{R}[t]$ with $\deg(Q) = r - 1$ and $0 < \rho < 1$ such that*

$$\mathrm{d}(f^n) = \lambda^n\big(Q(n) + \mathcal{O}(\rho^n)\big).$$

2) There exists a finite positive constant C such that for all $n \in \mathbb{N}$,

$$\frac{\mathrm{d}(f^{n+1}) - \lambda\,\mathrm{d}(f^n)}{\mathrm{d}(f^n)} \le \frac{C}{n^2} \qquad \textit{and} \qquad \sum_{j=0}^{n} \mathrm{d}(f^j) \le C\,\mathrm{d}(f^n).$$

Proof. Part 1) follows implicitly from the proof of Theorem 4.2 in [16]. We even know from the last work that $r = 1$ or $r = 2$. Part 2) is an immediate consequence of Part 1). $\square$

In this section we make the following convention: $\mathrm{d}(f^n) = 0$ for all $n < 0$.

Lemma 3.2. *The following identity holds*

$$F_n = \begin{cases} F_{n-1} \circ F, & n = 1, \dots, n_0, \\ \frac{F_{n-1} \circ F}{H^{\mathrm{d}(f^{n-n_0-1})}}, & n > n_0. \end{cases} \tag{3.3}$$

Moreover for all currents $T \in \mathcal{C}^+(\mathbb{P}^k)$,

$$(f^n)^* T = \begin{cases} f^*((f^{n-1})^* T), & n = 1, \dots, n_0, \\ f^*((f^{n-1})^* T) - \|T\| \cdot \mathrm{d}(f^{n-n_0-1}) \cdot [H = 0], & n > n_0. \end{cases} \tag{3.4}$$

Proof. We only need to prove identity (3.3) since identity (3.4) is an immediate consequence of (3.3) using (2.2)–(2.3). Next, observe that by the hypothesis on f, (3.3) is true for $n = 1, \dots, n_0$. Suppose (3.3) true for n, we need to prove it for $n+1$.

We have that

$$\begin{aligned} F_n \circ F &= \frac{F \circ F_{n-1} \circ F}{H \circ F_{n-n_0-1} \circ F} = \frac{F(F_{n-1} \circ F)}{H(F_{n-n_0-1} \circ F)} \\ &= \frac{F(H^{\mathrm{d}(f^{n-n_0-1})} F_n)}{H(H^{\mathrm{d}(f^{n-2n_0-1})} \cdot F_{n-n_0})} = \frac{H^{\mathrm{d}(f) \cdot \mathrm{d}(f^{n-n_0-1})} \cdot F \circ F_n}{H^{h \cdot \mathrm{d}(f^{n-2n_0-1})} \cdot H \circ F_{n-n_0}} \\ &= H^{\mathrm{d}(f) \cdot \mathrm{d}(f^{n-n_0-1}) - h \cdot \mathrm{d}(f^{n-2n_0-1})} F_{n+1} = H^{\mathrm{d}(f^{n-n_0})} F_{n+1}, \end{aligned}$$

where the first equality follows from Definition 2, the third one from the hypothesis of induction, and the last one from identity (3.1). Hence, (3.3) is true for $n+1$. $\square$

Put, for $N \geq 1$,

$$\sigma_N := \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{\mathrm{d}(f^n)} (f^n)^* \omega.$$

Then (σ_N) is a sequence of positive closed currents of bidegree $(1,1)$ such that $\|\sigma_N\| = 1$. Therefore, we can subtract a convergent subsequence $(\sigma_{N_j}) : \sigma_{N_j} \to \sigma$. Here, σ is a positive closed currents of bidegree $(1,1)$ such that $\|\sigma\| = 1$.

Lemma 3.3. *The following functional equation holds*

$$f^* \sigma = \lambda \cdot \sigma + \frac{d - \lambda}{h} \cdot [H].$$

Proof. We have that

$$\begin{aligned}
f^*\sigma_N-\lambda\sigma_N &= \frac{1}{N}\sum_{n=0}^{N-1}\Big(\frac{f^*((f^n)^*\omega)}{\mathrm{d}(f^n)}-\frac{\lambda(f^{n+1})^*\omega}{\mathrm{d}(f^{n+1})}\Big)+\frac{1}{N}\Big(\lambda\frac{(f^N)^*\omega}{\mathrm{d}(f^N)}-\lambda\omega\Big)\\
&= \frac{1}{N}\sum_{n=0}^{N-1}\frac{f^*(f^n)^*\omega-(f^{n+1})^*\omega}{\mathrm{d}(f^n)}\\
&+ \Big(\frac{1}{N}\sum_{n=0}^{N-1}\frac{\mathrm{d}(f^{n+1})-\lambda\,\mathrm{d}(f^n)}{\mathrm{d}(f^n)}\cdot\frac{(f^{n+1})^*\omega}{\mathrm{d}(f^{n+1})}+\frac{\lambda}{N}\Big(\frac{(f^N)^*\omega}{\mathrm{d}(f^N)}-\omega\Big)\Big)\\
&\equiv I+II.
\end{aligned}$$

Applying Lemma 3.2 yields that

$$I=\Big(\frac{1}{N}\sum_{n=0}^{N-1}\frac{\mathrm{d}(f^{n-n_0})}{\mathrm{d}(f^n)}\Big)[H]. \tag{3.5}$$

Recall from (2.3) that $\|\frac{(f^n)^*\omega}{\mathrm{d}(f^n)}\|=1$, $n\geq 0$. Therefore,

$$\frac{\lambda}{N}\left(\Big\|\frac{(f^N)^*\omega}{\mathrm{d}(f^N)}\Big\|+\|\omega\|\right)\to 0\quad\text{as } N\to\infty.$$

On the other hand, applying the first estimate of Part 2) of Lemma 3.1 yields that

$$\frac{1}{N}\sum_{n=0}^{N-1}\frac{|\,\mathrm{d}(f^{n+1})-\lambda\,\mathrm{d}(f^n)|}{\mathrm{d}(f^n)}\cdot\Big\|\frac{(f^{n+1})^*\omega}{\mathrm{d}(f^{n+1})}\Big\|\leq\frac{C}{N}\sum_{n=0}^{N-1}\frac{1}{n^2}\to 0,\quad\text{as } N\to\infty.$$

Inserting the latter two estimates into the expression of (II), we obtain that $II\to 0$ as $N\to\infty$. This, combined with (3.5) implies that

$$f^*\sigma=\lambda\cdot\sigma+\mu[H]$$

for some $\mu\in\mathbb{R}$. By equating the mass of both sides in the last equation and using (2.3), the desired conclusion of the lemma follows. □

In virtue of Proposition 3.3, we can fix a potential Θ of σ such that

$$\Theta\circ F=\lambda\Theta+\frac{d-\lambda}{h}\log|H|. \tag{3.6}$$

Lemma 3.4.

$$\Theta\circ F_n=\begin{cases}\lambda^n\Theta+\frac{d-\lambda}{h}\cdot\sum\limits_{j=1}^{n}\lambda^{j-1}\log|H\circ F_{n-j}|, & n=1,\dots,n_0,\\ \lambda^n\Theta+\frac{d-\lambda}{h}\cdot\sum\limits_{j=1}^{n_0}\lambda^{j-1}\log|H\circ F_{n-j}|, & n>n_0.\end{cases}$$

Proof. We proceed by induction. For $n=1$ the above formula follows from (3.6).

Suppose that the above inductive formula is true for n. We need to show it for $n+1$. Observe that

$$\begin{aligned}
\Theta\circ F_{n+1} &= \Theta\circ F_n\circ F-\mathrm{d}(f^{n-n_0})\log|H|\\
&= \lambda^n\Theta\circ F+\frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\log|H\circ F_{n-j}\circ F|-\mathrm{d}(f^{n-n_0})\log|H|\\
&= \lambda^{n+1}\Theta+\frac{d-\lambda}{h}\cdot\lambda^n\log|H|-\mathrm{d}(f^{n-n_0})\log|H|\\
&+ \frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\log|H(H^{\mathrm{d}(f^{n-j-n_0})}\cdot F_{n-j+1})|\\
&= \lambda^{n+1}\Theta+\frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\log|H\circ F_{n-j+1}|\\
&+ \Big(\frac{d-\lambda}{h}\cdot\lambda^n+\frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}h\,\mathrm{d}(f^{n-j-n_0})-\mathrm{d}(f^{n-n_0})\Big)\log|H|\\
&= \lambda^{n+1}\Theta+\frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\log|H\circ F_{n-j+1}|\\
&+ \Big(\lambda^{n-n_0}+\frac{d-\lambda}{h}\cdot\sum_{j=1}^{n_0}\lambda^{j-1}h\,\mathrm{d}(f^{n-j-n_0})-\mathrm{d}(f^{n-n_0})\Big)\log|H|,
\end{aligned}$$

where the first equality follows from (3.3), the second one from the hypothesis of induction, the third one from (3.6) and (3.3), and the last one from (3.2). Therefore, the proof of the inductive formula will be complete for $n+1$ if one can show that for all $n\geq 0$, $S_n=0$, where

$$S_n:=\lambda^n+(d-\lambda)\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\,\mathrm{d}(f^{n-j})-\mathrm{d}(f^n).$$

It follows from (3.1)–(3.2) and the above formula for S_n that $S_n-dS_{n-1}+hS_{n-n_0-1}=0$ for all $n\geq n_0+1$. Hence, the proof will be complete if one can show that $S_n=0$ for $n=0,\ldots,n_0$. But the last assertion is equivalent to the identity

$$\lambda^n+(d-\lambda)\cdot\sum_{j=1}^{n_0}\lambda^{j-1}\,\mathrm{d}(f)^{n-j}=\mathrm{d}(f)^n,$$

which is clearly true by using the convention preceding Lemma 3.2. Hence, the proof is complete. □

The following elementary lemma is needed.

Lemma 3.5. *Let (X,σ,μ) be a measurable space and $(g_n)_{n=0}^\infty$, $(h_n)_{n=1}^\infty\subset L^1(X,\mu)$ two sequence of complex-valued functions with $\|h_n\|_{L^1(X)}\leq 1$, $n\geq 1$. Let $P(t):=t^{n_0}+\alpha_1t^{n_0-1}+\cdots+\alpha_{n_0}$ be a polynomial whose roots are of modulus strictly smaller*

than 1. *Let* $(\epsilon_n)_{n=n_0}^{\infty} \subset \mathbb{R}^+$ *be a sequence with* $\lim_{n\to\infty} \epsilon_n = 0$. *Let* $(\alpha_{1n})_{n_0}^{\infty}, \ldots, (\alpha_{n_0 n})_{n=1}^{\infty}$ *be* n_0 *sequences in* $\mathbb{C}$ *such that for all* $n \geq n_0$ *and* $1 \leq j \leq n_0$,

- $g_n + \alpha_{1n} g_{n-1} + \cdots + \alpha_{n_0 n} g_{n-n_0} = h_n$;
- $|\alpha_{jn} - \alpha_j| < \epsilon_n$.

Then $(g_n)_{n=0}^{\infty}$ *is bounded in* $L^1(X, \mu)$.

Proof. Let $t_1, \ldots, t_{n_0}$ be the roots of $P(t)$. Consider two cases.
Case $t_1, \ldots, t_{n_0}$ are distinct: we can check that if $\gamma_1, \ldots, \gamma_{n_0} \in \mathbb{C}$ such that $\sum_{j=1}^{n_0} \gamma_j \cdot \frac{P(t)}{t-t_j} \equiv 0$ then $\gamma_1 = \cdots = \gamma_{n_0} = 0$. Consequently, there exist $\gamma_1, \ldots, \gamma_{n_0} \in \mathbb{C}$ such that

$$\sum_{j=1}^{n_0} \gamma_j \cdot \frac{P(t)}{t-t_j} \equiv t^{n_0-1}. \tag{3.7}$$

Next, write

$$\frac{P(t)}{t-t_j} = t^{n_0-1} + \beta_{j1} t^{n_0-2} + \cdots + \beta_{j n_0 - 1}, \qquad j = 1, \ldots, n_0. \tag{3.8}$$

Put

$$f_{jn} := g_n + \beta_{j1} g_{n-1} + \cdots + \beta_{j n_0 - 1} g_{n-n_0+1}, \qquad n \geq n_0 - 1,\ 1 \leq j \leq n_0.$$

Hence, (3.7) becomes

$$\sum_{j=1}^{n_0} \gamma_j f_{jn} = g_{n-1}, \qquad n \geq n_0 - 1,\ 1 \leq j \leq n_0. \tag{3.9}$$

The formula for f_{jn}, the first • of the hypothesis and identity (3.8) together imply that

$$|f_{jn} - t_j f_{j,n-1}| \leq |h_n| + \epsilon_{n-1} |g_{n-1}| + \cdots + \epsilon_{n-n_0} |g_{n-n_0}|, \qquad n \geq n_0,\ 1 \leq j \leq n_0.$$

It follows from the last estimate and (3.9) that there is a finite positive constant C such that

$$M_n^{'} \leq \rho^{'} M_{n-1}^{'} + C(\epsilon_{n-1} + \cdots + \epsilon_{n-n_0})(M_n^{'} + \cdots + M_{n-n_0+1}^{'}) + \|h_n\|_{L^1(X)}, \qquad n \geq 2n_0,$$

where $M_n^{'} := \max\{\|f_{1n}\|_{L^1(X)}, \ldots, \|f_{n_0 n}\|_{L^1(X)}\}$ for all $n \geq n_0 - 1$, and $\rho^{'} := \max_{1 \leq j \leq n_0} |t_j|$. Observe that $0 < \rho^{'} < 1$ since $|t_j| < 1$.

Fix a constant $\rho : \rho^{'} < \rho < 1$. Using the above estimate for $M_n^{'}$ repeatedly and taking into account that $\lim_{n\to\infty} \epsilon_n = 0$, we may find a sufficiently large integer $N > n_0$ such that

$$M_n \leq \rho M_{n-1} + 2\Big(\sum_{j=1}^{n_0} \|h_{n n_0 - j}\|_{L^1(X)}\Big), \qquad n > N,$$

where $M_n := \max\{\ M_{n n_0 - 1}^{'}, \ldots, M_{n n_0 - n_0}^{'}\}$ for all $n \geq N$. Consequently,

$$M_n \leq \frac{2}{1-\rho} \cdot \sum_{k=0}^{n-N-1} \rho^k \Big(\sum_{j=1}^{n_0} \|h_{(n-k)n_0 - j}\|_{L^1(X)}\Big) + \rho^{n-N} M_N$$

for all $n \geq N$. This, combined with the hypothesis that $\|h_n\|_{L^1(X)} \leq 1$ for all $n \geq 1$, implies the existence of a finite positive constant M such that

$$\left\|f_{jn}\right\|_{L^1(X)} < M, \qquad n \geq N,\ 1 \leq j \leq n_0.$$

This, coupled with (3.9), gives the desired conclusion.

Case $t_1, \dots, t_{n_0}$ are not distinct: Let $t_1, \dots, t_r$ be all distinct roots of $P(t)$ with multiplicity $m_1, \dots, m_r$ respectively. We can choose $\gamma_{11}, \dots, \gamma_{1m_1}, \dots, \gamma_{r1}, \dots, \gamma_{rm_r} \in \mathbb{C}$ such that

$$\sum_{j=1}^{r} \frac{(\gamma_{j1} + \gamma_{j2} t + \cdots + \gamma_{jm_j} t^{m_j - 1}) P(t)}{(t - t_j)^{m_j}} \equiv t^{n_0 - 1}.$$

The remaining part of the proof follows along the same lines as in the previous case. □

Let us recall that a *qpsh function on* $\mathbb{P}^k$ is an upper semi-continuous function $\phi:\ \mathbb{P}^k \to [-\infty, \infty)$ which is locally given as the sum of a plurisubharmonic and a smooth function. The following estimate due to V. Guedj (see Proposition 1.3 in [15]) is needed.

Lemma 3.6. *There exists a positive finite constant C such that for all qpsh functions $\phi \leq 0$, $\mathrm{dd^c}\,\phi \geq -\omega$, and for all $n \in \mathbb{N}$,*

$$\int_{\mathbb{P}^k} (|\phi| \circ f^n)\omega^k \leq C \sum_{j=0}^{n} \mathrm{d}(f^j).$$

Now we arrive at

Proof of Part (i) of Main Theorem. Recall that r is the multiplicity of the root $\lambda_1(f)$ of $P(t)$. For all $n > n_0$ consider the functions defined on $\mathbb{P}^k$

$$h_n := \frac{1}{n^{r-1}} \Big(\Theta + \frac{d - \lambda}{h} \cdot \sum_{j=1}^{n_0} \frac{1}{\lambda^{n-j+1}} \log \Big| H\big(\frac{F_{n-j}}{\|F_{n-j}\|}\big) \Big| - \frac{1}{\lambda^n} \Theta\big(\frac{F_n}{\|F_n\|}\big) \Big).$$

By Lemma 3.4, we have that

$$\frac{\log \|F_n\|}{n^{r-1}\lambda^n} - \frac{d - \lambda}{\lambda} \cdot \sum_{j=1}^{n_0} \frac{\log \|F_{n-j}\|}{n^{r-1}\lambda^{n-j}} = h_n. \tag{3.10}$$

On the one hand, we know from Part 1) of Lemma 3.1 that $\mathrm{d}(f^n) \approx n^{r-1}\lambda^n$. On the other hand, since Θ and $\log |H|$ are plurisubharmonic, an application of Lemma 3.6 and the second estimate of Part 2) of Lemma 3.1 gives that $\|h_n\|_{L(\mathbb{P}^k, \omega^k)} < C$ for a finite constant C independent of n. Moreover, the polynomial $t^{n_0} - \frac{d-\lambda}{\lambda} \cdot \sum\limits_{j=1}^{n_0} t^{n_0 - j}$ is equal to $\frac{1}{\lambda_0^n} \frac{P(\lambda t)}{\lambda t - \lambda}$ by using the identity $P(\lambda t) = P(\lambda t) - P(\lambda)$. Therefore, in virtue of (3.2) all roots of $t^{n_0} - \frac{d-\lambda}{\lambda} \cdot \sum\limits_{j=1}^{n_0} t^{n_0 - j}$ are of modulus strictly smaller than 1. Hence, we are in the position to apply Lemma 3.5 to the relations (3.10) with $\alpha_j := -\frac{d-\lambda}{\lambda}$ and

$\alpha_{jn} := -\frac{d-\lambda}{\lambda}\cdot\frac{(n-j)^{r-1}}{n^{r-1}}$, $1\leq j\leq n_0$. Consequently, it follows that $\frac{\log\|F_n\|}{n^{r-1}\lambda^n}$ is locally uniformly bounded in $L^1(\mathbb{C}^{k+1})$-norm. This proves Part (i).

Proof of Part (ii). Using identity (3.3) we have

$$\begin{aligned}\frac{\log\|F_n\circ F\|}{\mathrm{d}(f^n)} &= \frac{\log\|F_{n+1}\|}{\mathrm{d}(f^n)}+\frac{\mathrm{d}(f^{n-n_0})\log|H|}{\mathrm{d}(f^n)}\\ &= \lambda\cdot\frac{\log\|F_{n+1}\|}{\mathrm{d}(f^{n+1})}+\Big(\frac{\mathrm{d}(f^{n+1})}{\mathrm{d}(f^n)}-\lambda\Big)\frac{\log\|F_{n+1}\|}{\mathrm{d}(f^{n+1})}+\frac{\mathrm{d}(f^{n-n_0})\log|H|}{\mathrm{d}(f^n)}.\end{aligned}$$

Now take the $(\limsup_{n\to\infty})^*$ of both sides of the above identity. By Part (i), the left hand side is then $u\circ F$ and the first term of the right hand side is $\lambda\cdot u$. The second term of the right hand side is 0 by using the first estimate of Part 2) of Lemma 3.1 and the fact already proved in Part (i) that $\frac{\log\|F_n\|}{\mathrm{d}(f^n)}$ is locally uniformly bounded in $L^1(\mathbb{C}^{k+1})$-norm. The last term of the right hand side converges to $\frac{1}{\lambda^{n_0}}\cdot\log|H|$ using Part 1) of Lemma 3.1: $\mathrm{d}(f^n)\approx n^{r-1}\lambda^n$. In summary, we have shown that

$$u\circ F=\lambda\cdot u+\frac{1}{\lambda^{n_0}}\cdot\log|H|=\lambda\cdot u+\frac{d-\lambda}{h}\cdot\log|H|,$$

where the last equality follows from equation (3.2). This proves (ii).

Proof of Part (iii). Let $p\in U$, where U is an open set contained in the Fatou set. Shrinking U if necessary, we may assume that a subsequence f^{n_j} converges in U to a holomorphic map h and that $f^{n_j}(U)\subset\{z_0=1,|z_j|<2\}$. We can then write

$$\frac{\log\|F_{n_j}\|}{\mathrm{d}(f^{n_j})}=\frac{\log\|(F_{n_j})_0\|}{\mathrm{d}(f^{n_j})}+\frac{1}{\mathrm{d}(f^{n_j})}\log\|(1,A_j^1,\ldots,A_j^k\|.$$

The last term converges uniformly to 0, and the first term is pluriharmonic. Hence, using Part (i) the function u is pluriharmonic on U, and U does not intersect the support of T.

4. Examples

First we recall the result from our previous work [16].

4.1. A sufficient condition for QAS self-maps.

In [14] Fornæss and Sibony establish the following definition.

Definition 4.1. *A hypersurface $\mathcal{H}\subset\mathbb{P}^k$ is said to be a degree lowering hypersurface of f if, for some (smallest) $n\geq 1$, $f^n(\mathcal{H})\subset\mathcal{I}(f)$. The integer n is then called the height of $\mathcal{H}$.*

The following (see Proposition 3.2 in [16]) gives us the structure of a non AS self-map.

Proposition 4.2. *Let f be a meromorphic self-map of $\mathbb{P}^k$. Then there are exactly an integer $M\geq 0$, M degree lowering hypersurfaces $\mathcal{H}_j$ with height n_j, $j=1,\ldots,M$, satisfying the following properties:*
(i) all the numbers n_j , $j=1,\ldots,M$, are pairwise different;
(ii) $\operatorname{codim}(f^m(\mathcal{H}_j))>1$ *for $m=1,\ldots,n_j$, and $j=1,\ldots,M$;*

(iii) for any degree lowering irreducible hypersurface $\mathcal{H}$ of f, there are integers $n \geq 0$ and $1 \leq j \leq M$ such that $f^n(\mathcal{H})$ is a hypersurface and $f^n(\mathcal{H}) \subset \mathcal{H}_j$

In particular, f is AS if and only if $M = 0$.

Definition 4.3. *Under the hypothesis and the notation of Proposition 4.2, for every $j = 1, \ldots, M$, $\mathcal{H}_j$ is called the primitive degree lowering hypersurface of f with the height n_j.*

We are now able to state a sufficient criterion for QAS maps (see Main Theorem in [16]).

Theorem 4.4. *A meromorphic self-map f of $\mathbb{P}^k$ is QAS if it satisfies the following properties (i)–(iii):*

- (i) *there is only one primitive degree lowering hypersurface (let $\mathcal{H}_0$ be this hypersurface and let n_0 be its height);*
- (ii) *for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$ and every $m = 1, \ldots, n_0$, $f^m(\mathcal{H}) \not\subset \mathcal{H}_0$;*
- (iii) *for every irreducible component $\mathcal{H}$ of $\mathcal{H}_0$, one of the following two conditions holds*
  $(iii)_1$ $f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all $m \geq n_0 + 1$,
  $(iii)_2$ there is an $m_0 \geq n_0$ such that $f^{m_0+1}(\mathcal{H})$ is a hypersurface and $f^m(\mathcal{H}) \not\subset \mathcal{I}(f)$ for all m verifying $n_0 + 1 \leq m \leq m_0$.

It is worthy to remark that Proposition 4.2 allows us to check if a map is QAS. The remaining of this section is devoted to the study of new parameterized families of QAS maps in $\mathbb{P}^2$.

4.2. **New family of QAS self-maps of $\mathbb{P}^2$.** Let P be a (not necessarily irreducible) homogeneous polynomial in $\mathbb{C}^3$. Let Q_1, Q_2 and Q_3 be (not necessarily irreducible) homogeneous polynomials in $\mathbb{C}^3$ of the same degree. Let R be a (not necessarily irreducible) homogeneous polynomial in $\mathbb{C}^3$ such that $\deg(R) = \deg(P) + \deg(Q_1)$ and that

$$P(1,1,1)Q_j(1,1,1) = R(1,1,1) \neq 0, \qquad j = 1,2,3. \tag{4.1}$$

Suppose for the moment that $PQ_1 - R,\ PQ_2 - R,\ PQ_3 - R$ have no nontrivial common factor, we are able to define a dominant meromorphic map of $\mathbb{P}^2$

$$f([z:w:t]) := [PQ_1 - R : PQ_2 - R : PQ_3 - R]\,. \tag{4.2}$$

It can be checked that for every $(a,b,c) \in \mathbb{C}^3 \setminus \{0\}$ with $a+b+c = 0$, the hypersurface $\{aQ_1 + bQ_2 + cQ_3 = 0\}$ is sent by f into the complex line $\{az + bw + ct = 0\}$.

Proposition 4.5. *Suppose that for all $(a,b,c) \in \mathbb{C}^3 \setminus \{0\}$ with $a+b+c = 0$, every irreducible component of the hypersurface $\{aQ_1 + bQ_2 + cQ_3\}$ is sent by f onto a hypersurface. Suppose in addition that every irreducible component of the hypersurface $\{P = 0\}$ is sent by f^2 onto a hypersurface and that if $\mathcal{G}$ is an irreducible hypersurface such that $f(\mathcal{G}) = [1:1:1]$ then $\mathcal{G} \subset \{P = 0\}$. Then f satisfies the properties (i)–(iii) listed in Theorem 4.4, in particular, f is QAS.*

Proof. First observe by (4.2) and (4.1) that the hypersurface $\{P = 0\}$ is sent by f to the point $[1 : 1 : 1] \in \mathcal{I}(f)$. We will show that there is no irreducible degree lowering hypersurface other than the components of $\{P = 0\}$. To do this suppose, in order to get a contradiction, that G is an irreducible homogeneous polynomial in $\mathbb{C}^3$ such that $\mathcal{G} \not\subset \{P = 0\}$ and that $f(\mathcal{G})$ is a point $[a : b : c] \in \mathbb{P}^2$, where $\mathcal{G}$ is the hypersurface $\{G = 0\}$ in $\mathbb{P}^2$. Suppose, without loss of generality, that $a \neq 0$. We deduce from (4.2) and the equality $f(\mathcal{G}) = [a : b : c]$ that G divides both polynomials $P(bQ_1 - aQ_2) - (b - a)R$ and $P(cQ_1 - aQ_3) - (c - a)R$. Hence, G divides the polynomial

$$P \cdot \Big((c-a)(bQ_1 - aQ_2) - (b-a)(cQ_1 - aQ_3)\Big).$$

Since $\mathcal{G} \not\subset \{P = 0\}$ and G is irreducible, we see that G divides the polynomial $(ac-ab)Q_1 + (a^2 - ac)Q_2 + (ab - a^2)Q_3$. Since $a \neq 0$, we deduce from the first hypothesis that either $f(\mathcal{G})$ is a hypersurface or $a = b = c$. The former case contradicts the assumption that $f(\mathcal{G})$ is a point $[a : b : c] \in \mathbb{P}^2$. The latter case implies that $f(\mathcal{G}) = [1 : 1 : 1]$, which, by the third hypothesis, gives that $\mathcal{G} \subset \{P = 0\}$, which contradicts our assumption.

We have shown that $\{P = 0\}$ is the unique primitive degree lowering hypersurface and its height is 1. Since by (4.1) $[1 : 1 : 1] \notin \{P = 0\}$, and every irreducible component of the hypersurface $\{P = 0\}$ is sent by f^2 onto a hypersurface, it follows that f satisfies (i)–(iii) of Theorem 4.4. □

Now we will discuss when the hypotheses of Proposition 4.5 are fulfilled.

Corollary 4.6. *Suppose that $Q_2 - Q_1$, $Q_3 - Q_1$ are coprime and that P, R are coprime. Then $PQ_1 - R$, $PQ_2 - R$, $PQ_3 - R$ have no nontrivial common factor. Moreover, for every irreducible hypersurface $\mathcal{G}$ with $f(\mathcal{G}) = [1 : 1 : 1]$, we have $\mathcal{G} \subset \{P = 0\}$. Here f is defined by (4.2).*

Proof. It is left to the interested reader as an exercise. □

Corollary 4.7. *Suppose that the pre-image of the point $[1 : 1 : 1]$ by the map $\mathbb{P}^2 \ni [z : w : t] \mapsto [Q_1 : Q_2 : Q_3]$ is a set of finite points and that for every $[z : w : t] \in \{P = 0\} \cap \{R = 0\}$ and every $(a, b, c) \in \mathbb{C}^3 \setminus \{0\}$ with $a+b+c = 0$, we have $(aQ_1 + bQ_2 + cQ_3)(z, w, t) \neq 0$. Suppose in addition that for every $(a, b, c) \in \mathbb{C}^3 \setminus \{0\}$ with $a + b + c = 0$, two polynomials P and $aQ_1 + bQ_2 + cQ_3$ are coprime. Then every irreducible component of the hypersurface $\{aQ_1 + bQ_2 + cQ_3\}$ is sent by f onto a hypersurface.*

Proof. Suppose in order to get a contradiction that $\mathcal{G}$ is an irreducible component of the hypersurface $\{aQ_1 + bQ_2 + cQ_3\}$ and $f(\mathcal{G})$ is a point $p \in \mathbb{P}^2$, where $(a, b, c) \in \mathbb{C}^3 \setminus \{0\}$ with $a + b + c = 0$. Using the explicit formula (4.2), the second hypothesis ensures that there exists

$$[z_0 : w_0 : t_0] \in \big(\{aQ_1 + bQ_2 + cQ_3\} \cap \{P = 0\}\big) \setminus \mathcal{I}(f).$$

Consequently, we get $p = f([z_0 : w_0 : t_0]) = [1 : 1 : 1]$. This implies that either the map $\mathbb{P}^2 \ni [z : w : t] \mapsto [Q_1 : Q_2 : Q_3]$ sends $\mathcal{G}$ to the point $[1 : 1 : 1]$ or

$\mathcal{G} \subset \{P = 0\}$. But the former case contradicts the first hypothesis whereas the latter case contradicts the third hypothesis. $\square$

Corollary 4.8. *Suppose that for every $(a, b, c) \in \mathbb{C}^3 \setminus \{0\}$, two polynomials P and $aQ_1 + bQ_2 + cQ_3$ are coprime. Suppose in addition that the 3×3 matrix whose j-line is*

$$\Big(\frac{\partial(PQ_j - R)}{\partial z}(1,1,1) \quad \frac{\partial(PQ_j - R)}{\partial w}(1,1,1) \quad \frac{\partial(PQ_j - R)}{\partial t}(1,1,1) \Big)$$

has the rank ≥ 2. Then every irreducible component of the hypersurface $\{P = 0\}$ is sent by f^2 onto a hypersurface.

Proof. Let $F : \mathbb{C}^3 \to \mathbb{C}^3$ be given by

$$F = (F_1, F_2, F_3) := (PQ_1 - R, PQ_2 - R, PQ_3 - R).$$

Then a straightforward computation shows that the j-component of $\frac{F \circ F}{P}$ $(1 \leq j \leq 3)$ has the form

$$\frac{\partial(PQ_j - R)}{\partial z}(F) \cdot Q_1 + \frac{\partial(PQ_j - R)}{\partial w}(F) \cdot Q_2 + \frac{\partial(PQ_j - R)}{\partial t}(F) \cdot Q_3 + \mathcal{O}(P),$$

where $\mathcal{O}(P)$ is a polynomial which can be factored by P. Observe that the proof of the corollary will be complete if we can show that for any fixed irreducible divisor S of P, the image of $[S = 0]$ by $\frac{F \circ F}{P}$ $(1 \leq j \leq 3)$ is a curve. Using the above formula, this task is reduced to show that the (not necessarily dominant) rational map of $\mathbb{P}^2$ whose j-component is

$$\frac{\partial(PQ_j - R)}{\partial z}(1,1,1) \cdot Q_1 + \frac{\partial(PQ_j - R)}{\partial w}(1,1,1) \cdot Q_2 + \frac{\partial(PQ_j - R)}{\partial t}(1,1,1) \cdot Q_3,$$

does not map $[S = 0]$ to a point. But this is always satisfied taking into account the hypothesis. $\square$

Now we fix the degrees of P and Q_1. Using the above corollaries, we see easily that with a generic choice of the coefficients of R, P, Q_1, Q_2, Q_3 such that relation (4.1) holds, the hypotheses of Corollary 4.6, 4.7 and 4.8 are fulfilled. We thus obtain a family of non AS but QAS self-maps. The characteristic polynomial of maps in this family is (see (3.2))

$$P(t) := t^2 - (\deg(P) + \deg(Q_1))t + \deg(P).$$

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